are seen to have a repeating cycle. However, the inclusion of friction induces a decaying mechanism that results in a better agreement with the experimental results. (Compare the dashed and the dashed-dotted lines.)

Conclusions

The head-on reflection of a planar shock wave from a rubber wall experiencing a uniaxial strain loading mode that was investigated numerically during a previous study[1] was compared with experimental results that were obtained recently.

It has been demonstrated that Kistler piezoelectric pressure transducers could be used to record stresses in rubberlike materials, although they originally were designed to measure pressures in fluids.

In view of the previous remark, the comparison of the actual experimental results with the numerical simulations revealed a very good agreement as far as the durations of the stress pulses are involved and fairly good agreements as far as the shapes and peak values of the stress pulses are concerned.

Finally, the conclusions from the numerical investigation[1] that rubberlike materials cannot be used to reduce head-on reflecting shock wave loads on structure have been verified experimentally. Both the numercial and the experimental investigations clearly indicate that the presence of the rubber results in a significant amplification of the pressure acting on the endwall of the shock tube. Consequently, experimental setups similar to that shown in Fig. 1 could be used as pressure amplifiers.

Acknowledgment

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Large Displacement Axisymmetric Element For Nonaxisymmetric Deformation

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I. Introduction

LARGE deformation, three-dimensional analyses of axisymmetrical structures can be costly in spite of the relatively simple geometry. One approach to efficiently analyzing such problems is to use axisymmetric elements that are formulated to allow for nonaxisymmetric deformations. A cylindrical coordinate system in the context of a total Lagrangian formulation seems appropriate for these elements.

Although many programs use cylindrical, spherical, or local coordinate systems for input convenience, the underlying formulation is usually based on Cartesian reference systems.[1–3] Although significant efforts have been devoted to cylindrical formulations, most efforts have been restricted to small displacements[4,5] or shell elements.[6–11]

This Note discusses a Lagrangian finite element formulation in general orthogonal curvilinear coordinate systems, including the basic components describing Lagrangian strains and strain increments and the procedure for integrating these into the virtual work equation. This generalized formulation is applied to a solid cylindrical finite element that is demonstrated at the end of this Note. Although developed to address structural problems in the oil industry, the formulation developed herein is general with much broader application potential.

II. Lagrangian Strains in Curvilinear Coordinates

Lagrangian formulations use the Green-Lagrange strains because they have the desirable characteristic of remaining invariant under rigid-body rotation. Many authors express Green-Lagrange strains in terms of the deformation gradient in general curvilinear or orthogonal curvilinear coordinate systems.[1,12–14] The development that follows is based largely on Malvern's discussion.[12] In orthogonal coordinate systems, the covariant and contravariant components are coincident, and so only one component type needs to be considered. Therefore, the convention of summation for repeated subscripts is used in this Note.

The spatial components of the material vector may be expressed in terms of the material components through the deformation gradient tensor F_{km}:

$$\mathrm{d}s_k = \frac{h^k}{h^m}\left(\frac{\partial x_k}{\partial X_m}\right)\mathrm{d}S_m = F_{km}\,\mathrm{d}S_m \tag{1}$$

This can be substituted into the expression for the Green-Lagrange strains, giving

$$E_{ij} = \frac{1}{2}\left[\frac{(h^k)^2}{H^i H^j}\frac{\partial x_k}{\partial X_i}\frac{\partial x_k}{\partial X_j} - \delta_{ij}\right] \tag{2}$$

In finite element applications the strains must be expressed in terms of the displacement field. This requires appropriate displacement measures to be defined. Although Cartesian scale factors are independent of location and displacements, in curvilinear coordinate systems location-dependent scale functions add considerable complexity. At this point most authors simplify their discussions to infinitesimal strain formulations,[12,13] so that physical displacements can be used as field variables, and the scale functions can be assumed to be constant.

Defining physical displacements that include deformation-dependent scale functions is more difficult and unnecessary. Instead, coordinate displacements can be used as the field variables. This approach was used by Truesdell and Toupin[14] for Lagrangian strains in general curvilinear coordinate systems; however, it has not previously been used in incremental form for a finite element formulation. The displacements are defined simply as

$$u_i = x_i - X_i \tag{3}$$

where $i = r, \theta, z$ in a cylindrical system.

The strains can then be expressed in terms of the displacement field and scale functions:

$$E_{ij} = \frac{1}{2H^i H^j}\left(h^{j^2} u_{j,i} + h^{i^2} u_{i,j} + h^{k^2} u_{k,j} u_{k,i} - h^{k^2}\delta_{kj}\delta_{ki} - \delta_{ij}\right) \tag{4}$$

The displacement gradient components in the first three terms are similar to the usual Cartesian expressions. The last two terms

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are pure scale factor components describing, for example, the hoop strain produced by pure radial expansion.

In incremental form, the strains are separated into parts corresponding to the order of displacement increment:

$$\Delta E_{ij} = \Delta\varepsilon_{ij} + \Delta\varepsilon'_{ij} + \Delta\varepsilon''_{ij} + \Delta\varepsilon'''_{ij} + \Delta\varepsilon''''_{ij} \quad (5)$$

in which

$$2H^iH^j\Delta\varepsilon_{ij} = \left(h^{i^2}\Delta u_{j,i} + h^{j^2}\Delta u_{i,j}\right) + (2h^k\Delta h^k\delta_{ki}\delta_{kj}) \quad (6a)$$

$$2H^iH^j\Delta\varepsilon'_{ij} = h^{k^2}(u_{k,j}\Delta u_{k,i} + \Delta u_{k,j}u_{k,i})$$

$$+ 2(h^j\Delta h^j u_{j,i} + h^i\Delta h^i u_{i,j} + h^k\Delta h^k u_{k,j}u_{k,i}) \quad (6b)$$

$$2H^iH^j\Delta\varepsilon''_{ij} = h^{k^2}\Delta u_{k,j}\Delta u_{k,i} + \Delta h^{j^2}u_{j,i} + \Delta h^{i^2}u_{i,j}$$

$$+ \Delta h^{k^2}(u_{k,j}u_{k,i} + \delta_{kj}\delta_{ki}) + 2[h^j\Delta h^j\Delta u_{j,i} + h^i\Delta h^i\Delta u_{i,j}$$

$$+ h^k\Delta h^k(u_{k,j}\Delta u_{k,i} + u_{k,i}\Delta u_{k,j})] \quad (6c)$$

The infinitesimal strain component $\Delta\varepsilon$ does not depend on the displacement gradient. The second term $\Delta\varepsilon'$ is also a first-order strain increment component but is nonlinear because it depends on the displacement gradients at time t. The remainder of the terms are second-, third-, and fourth-order strain increment components. The first- and second-order terms are used in the finite element stiffness formulation. Higher order terms are not shown because they are ignored in the linearization of the equations and do not arise in the equilibrium equations. Thus, the error introduced by the linearizing assumption can be eliminated by equilibrium iterations.

III. Finite Element Equations

The linear stiffness matrix is integrated using the operator matrix that relates strains to the displacement field. The first-order strain increments are therefore given by

$$\{\Delta\varepsilon\} = [b]\{\Delta u\}, \qquad \{\Delta\varepsilon'\} = [b']\{\Delta u\} \quad (7)$$

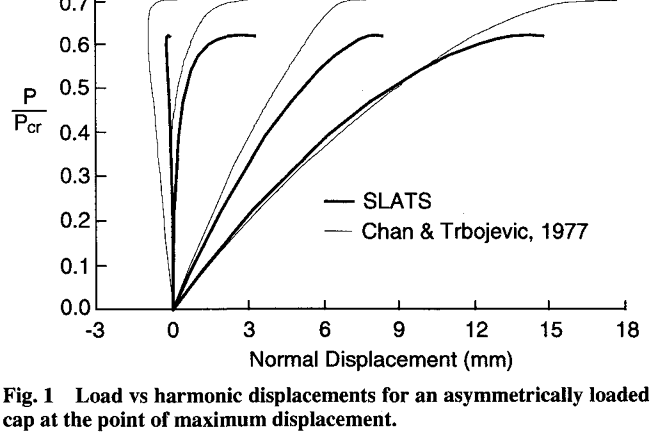

Fig. 1 Load vs harmonic displacements for an asymmetrically loaded cap at the point of maximum displacement.

where $[b]$ and $[b']$ are differential operator matrices, $\{\Delta\varepsilon\}$ and $\{\Delta\varepsilon'\}$ are vectors containing the first-order strain components, and $\{\Delta u\}$ is a vector of displacement field functions.

Although the displacement gradient components of $[b]$ and $[b']$ are simply scaled versions of those of Bathe,[1] the scale function components are not seen in the common Cartesian formulations. These two components can be shown separately:

$$[b] = [d] + [h], \qquad [b'] = [d'] + [h'] \quad (8)$$

where $[d]$ and $[d']$ are the displacement gradient components, and $[h]$ and $[h']$ are the scale function components. Use of the $[b]$ matrix in the stiffness matrix and equilibrium evaluation is shown by Bathe.[1]

The second-order strain increments lead to the geometric or nonlinear stiffness matrix. These increments can be separated into subsets in the same manner used with $\Delta\varepsilon$ and $\Delta\varepsilon'$. However, three subsets can be defined here: second-order displacement gradient components, second-order scale function components, and cross terms. Three components of the geometric stiffness operators $[k_g]$ matrix are correspondingly produced:

$$[k_g] = [k^D] + [k^H] + [k^{DH}] \quad (9)$$

IV. Sample Application

This finite element formulation was applied in an axisymmetric element in a cylindrical coordinate system with coordinates r, z, and θ and corresponding scale functions of 1, 1, and r. The hybrid displacement field of Zienkiewicz[2] with Fourier decomposition was used with coordinate rather than physical displacement components:

$$u_i = \sum_{n=1}^{N_p} P_n(R,Z) \sum_{f=0}^{N_f} u_{ic}^{nf}\cos(n\theta) + u_{is}^{nf}\sin(n\theta) \quad (10)$$

where $P_n(R,Z)$ are the usual polynomial interpolation functions, N_p and N_f are the number of nodes and harmonics, and $u_{i(c|s)}^{nf}$ are nodal harmonic displacement amplitudes for node n and Fourier number f, with c and s denoting cosine and sine terms, respectively. The element is explored in detail by Kaiser,[15] and two sample results are presented here.

The axisymmetric cap under an asymmetric load shown in Fig. 1 is discussed by Chan and Firmin[8] and Chan and Trbojevic[9,10] in the context of a Fourier-based finite element analysis. Figure 1 compares the harmonic displacement amplitudes at the maximum displacement location for the solid formulation with the shell-mixed formulation of Chan and Trbojevic. Although the initial stiffnesses compare well, the collapse load given by Chan and Trbojevic's formulation is significantly higher. It is suspected that small displacement kinematic assumptions, the stress distribution modeled through the shell thickness, and the coarse axisymmetric mesh employed by Chan and Trbojevic contribute to this higher buckling load. The results also illustrate the harmonic coupling in nonlinear problems. Although there is no loading component in the 2Θ harmonic, there is a significant 2Θ displacement component. The coupled displacement becomes particularly pronounced as the buckling load is approached.

Figure 2 shows displacement results for a tube column modeled with a slightly eccentric axial load compared with the linearized beam-column analytical solution. The lateral displacements remain relatively small until the Euler buckling load is approached, at which point the displacements increase drastically at the end. Transverse displacements vs axial load curves are plotted for analyses using different numbers of Fourier terms. The results show excellent agreement with the analytical solution below the collapse load, demonstrating the effectiveness of the formulation. However, limitations in the element displacement field cause the solution to diverge as the lateral displacements increase. The effective range of displacement can be improved by increasing the number of harmonics modeled. However, a more refined element displacement

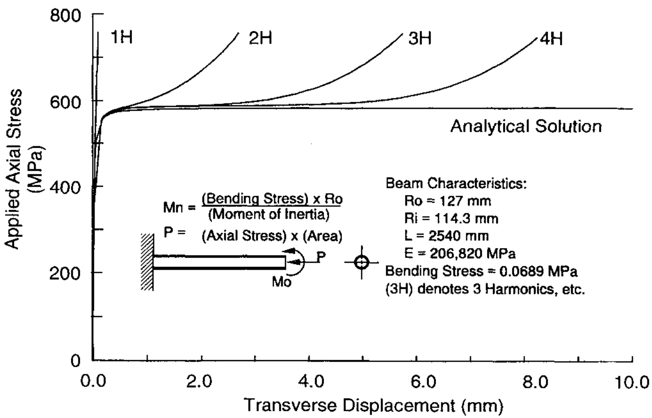

Fig. 2 Elastic beam column test case.

field should be defined, considering coordinate displacements to make such an element more generally applicable.

V. Conclusions

A new finite element formulation has been developed in terms of general orthogonal curvilinear coordinate systems. The formulation has been demonstrated by application to a basic cylindrical element using cylindrical displacement components and Fourier decomposition for the circumferential displacement field. The element accurately models nonlinear behavior within the effective range of element displacements. Development of a refined element displacement field is the focus of continued research to develop a general use axisymmetric element to model nonlinear three-dimensional behavior.

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